Chapter 3.10: Related Rates

Related Rates Idea

Idea: If some variables are related by an equation, their rates of change are also related by an equation.

Goal: Given some rates of change, find others.

Think of everything as being functions of time, i.e., rates are with respect to time.

Solving process

- Read problem carefully!
- Label everything: assign symbols to all quantities that are functions of time.
- List the given information, using the new symbols if needed.
- Draw a picture (if possible).
- Write an equation that relates the various quantities of the problem that are changing.
- Use implicit differentiation to take derivative of everything with respect to time.
- Plug in what is known, solve for what is wanted.

Example

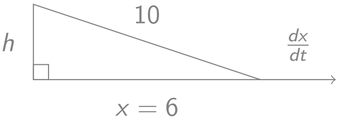
A particle is moving along the (implicit) curve $x^3 + y^3 = 9xy$. Given that when the particle is at $(2, 4)$ that $\frac{dx}{dt} = -2$, find $\frac{dy}{dt}$.

Would be cool to draw a figure but... Anyway, we know $x = 2$, $y = 4$ $\frac{dx}{dt} = -2$. We also know relation between x and y to be $x^3 + y^3 = 9xy$. Now, we write both x and y as functions of t and then take the implicit derivative.

$$\begin{aligned}
x^3 + y^3 &= 9xy \\
\frac{d}{dt}\left[x^3 + y^3\right] &= \frac{d}{dt}\left[9xy\right] \\
3x^2 \cdot \frac{dx}{dt} + 3y^2 \cdot \frac{dy}{dt} &= 9\frac{dx}{dt} \cdot y + 9x \cdot \frac{dy}{dt} \\
3 \cdot 2^2 \cdot (-2) + 3 \cdot 4^2 \cdot \frac{dy}{dt} &= 9 \cdot (-2) \cdot 4 + 9 \cdot 2 \cdot \frac{dy}{dt} \\
-4 + 8 \cdot \frac{dy}{dt} &= -12 + 3\frac{dy}{dt} \\
\frac{dy}{dt} &= -\frac{8}{5}
\end{aligned}$$

A 10 foot ladder is leaning against a wall and the bottom of the ladder is currently 6 feet from the wall and sliding away from the wall at rate of $2\frac{\text{ft}}{\text{min}}$. How quickly is the other end of the ladder moving down the wall?

Ladder length is 10 ft. Bottom is $x = 6$ ft from the wall. Slide away $\frac{dx}{dt} = 2\frac{\text{ft}}{\text{min}}$. Height on the wall h in ft. What is change of h, that is $\frac{dh}{dt}$?

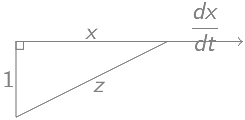


From the picture we get $h^2 + x^2 = 10^2$. So when $x = 6$, we get $h = \sqrt{100 - 36} = \sqrt{64} = 8$. Now the derivatives:

$$\frac{d}{dt}\left[h^2 + x^2\right] = \frac{d}{dt}[100]$$
$$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$$
$$16\frac{dh}{dt} + 24 = 0$$
$$\frac{dh}{dt} = -\frac{3}{2}$$

A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 6 mi away from the station.

Let x be the horizontal position of the plane, in miles (mi), as a function of time t, in hours (h). Let z be the distance from the plane to the station, in miles (mi), as a function of time t, in hours (h). We are given that $\frac{dx}{dt} = 500$ and the altitude is 1 mi. We want $\frac{dz}{dt}$ when $z = 6$. Let's draw a picture:



From the picture, we have that $x^2 + 1 = z^2$. Differentiating: $2x\frac{dx}{dt} = 2z\frac{dz}{dt}$. Cleaning up: $x\frac{dx}{dt} = z\frac{dz}{dt}$. We want $\frac{dz}{dt}$ when $z = 6$. We know that $\frac{dx}{dt} = 500$. Plugging in yields:

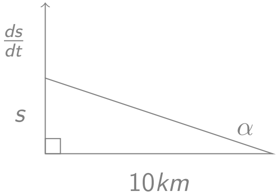
$$x(500) = (6)\frac{dz}{dt}$$

What about x? We know that $x^2 + 1 = z^2$. When $z = 6$, we get that $x^2 + 1 = 36$, thus $x = \sqrt{35}$. Hence

$$\frac{dz}{dt} = \frac{250\sqrt{35}}{3} \text{ mi/h}$$

You are 10 km away from a rocket launch. Close enough to see it but not so close to be instantly vaporized. You are filming the rocket as it goes straight up. Given that at the moment, your camera forms an angle of $\frac{\pi}{3}$ with the ground and that the angle is increasing at a rate of $0.2\frac{rad}{min}$, find the speed of the rocket.

The distance to base is 10 km, angle of camera is α rad, the position of the rocket is s in km. The velocity of rocket is $\frac{ds}{dt}$.



We have relation $\tan\alpha = \frac{s}{10}$

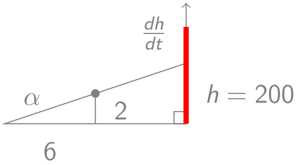
$$\frac{d}{dt}\left[10\tan\alpha\right] = \frac{d}{dt}\left[s\right]$$

$$10\frac{1}{(\cos\alpha)^2}\frac{d\alpha}{dt} = \frac{ds}{dt}$$

$$10\cdot\frac{1}{\frac{1}{4}}\cdot 0.2 = \frac{ds}{dt}$$

$$\frac{dh}{dt} = 8\frac{km}{min}$$

My wife and I were visiting Paris, and I wanted to have a photo of myself with Eiffel tower, which is 300m tall, where I would look as tall as the tower. We prepared for the photo. I raised my hand and it was just 2m high. My wife took the camera, stood 6m away from me and put the camera almost to the ground to make me look tall. Then she said, you appear to be 2/3 of the height of the tower and she started moving towards me at speed $\frac{1}{3}\frac{\text{m}}{\text{s}}$ to make me look as tall as the tower in the picture. How fast did I appear to be growing?



Notice my wife is 600m from the tower. Denote by s her distance from start. She starts at $s = 0$.

Her speed is $\frac{ds}{dt} = \frac{1}{3} m/s$. We get relation $\frac{2}{6-s} = \tan\alpha = \frac{h}{600-s}$. Hence $1200 - 2s = 6h - hs$. After taking derivative $-2\frac{ds}{dt} = 6\frac{dh}{dt} - \frac{ds}{dt}h - s\frac{dh}{dt}$. When known values are used $(200-2)\frac{ds}{dt} = 6\frac{dh}{dt}$. $33\frac{ds}{dt} = \frac{dh}{dt}$. Hence $\frac{dh}{dt} = 11\frac{m}{s}$. I grow so fast!